Exercise 41

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}[\ln(1+x^2)-\ln(1+x)]$$

Solution

Use the property of logarithms that allows a difference to be written as a quotient. Then multiply the numerator and denominator by the reciprocal of the highest power of x in the denominator.

$$\begin{aligned}
\lim_{x\to\infty}[\ln(1+x^2)-\ln(1+x)] &= \lim_{x\to\infty} \ln \frac{1+x^2}{1+x} \\
&= \lim_{x\to\infty} \ln \frac{1+x^2}{1+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \ln \frac{(1+x^2)\frac{1}{x}}{(1+x)\frac{1}{x}} \\
&= \lim_{x\to\infty} \ln \frac{\frac{1}{x}+x}{\frac{1}{x}+1} \\
&= \ln \frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}+x\right)}{\lim\limits_{x\to\infty}\left(\frac{1}{x}+1\right)} \\
&= \ln \frac{\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty} x}{\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty} 1} \\
&= \ln \frac{0+\infty}{0+1} \\
&= \ln\infty \\
&= \infty
\end{aligned}$$